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THE METHOD OF AXISYMMETRIC GENERALIZED ANALYTICAL FUNCTIONS IN THE ANALYSIS OF DYNAMIC PROCESSES*

V.F. PIVEN

A method for analysing various dynamic processes based on axisymmetric generalized analytical functions. Such functions satisfy a system of equations describing axisymmetric processes, which is canonical for two-dimensional dynamic processes in curvilinear layers with certain non-homogeneous behaviour of the characteristics and also for some kinds of anisotropy. The method, applied to some three-dimensional boundary-value seepage problems in a piecewise-homogeneous medium, produced solutions of these problems in a finite form.

1. Consider stationary and quasistationary physical processes described by a linear dynamic law and the dimensionless continuity equation /1/

$$\mathbf{V} = K\nabla\varphi, \quad \nabla\mathbf{V} = 0 \tag{1.1}$$

where $\mathbf{V}$ and φ are the velocity and the velocity potential of the process. For a homogeneous medium, K is a constant scalar quantity, for a non-homogeneous medium, it is a function of the coordinates and for an anisotropic medium, it is a symmetric second-rank tensor.

Plane-parallel processes are efficiently analysed using the method of analytical functions. The theory of these functions can be generalized by introducing some well-known assumptions /1-7/. For the axisymmetric case, there is a complete theory of functions that satisfy the following system of equations, which is obtained from (1.1) for $K = 1$ /2/:

$$u = \varphi_x = \psi_y/y, \; v = \varphi_y = -\psi_x/y \tag{1.2}$$

(ψ is the stream function). This system of equations describes various physical processes (seepage, heat conduction, electrical conduction, etc.). The most graphic physical interpretation of these equations is provided by the axisymmetric steady flow of an ideal incompressible fluid.

In Eqs.(1.2), the x and y axes are chosen in one of the half-planes through this axis of symmetry ox and u and v are the projections of the fluid velocity on these axes.

Eqs.(1.2) describe more than axisymmetric processes. They have a broader physical interpretation. Specifically, they may be regarded as equations of a process in a homogeneous layer whose thickness P is a linear function ($P = y$). Applying a conformal mapping defined by analytical functions to Eqs.(1.2), we transform them in the new variables x_1, y_1 to the form /2/

$$\varphi_{x_1} = \psi_{y_1}/P, \quad \varphi_{y_1} = -\psi_{x_1}/P \tag{1.3}$$

where $P = f(x_1, y_1)$ is a harmonic function and $f(x_1, y_1) = 0$ is the boundary of the region where the process evolves.

Eqs.(1.3) describe processes in layers with a harmonic variation of the thickness $f(x_1, y_1)$. If x_1, y_1 are interpreted as isothermal coordinates of the surface, then the layer is located on this surface.

The system of Eqs.(1.3) and therefore also (1.2) are obtained from Eqs.(1.1) written for two-dimensional processes in curvilinear layers of constant thickness with harmonic variation of the non-homogeneity, and also for a certain kind of anisotropy of the medium /1/.

Using the transformation formulas of /2/, we can associate axisymmetric processes to the processes described by Eqs.(1.3) for $P = f^n(x_1, y_1)$, where n is an odd integer.

Thus, the system of Eqs.(1.2) is canonical for various two-dimensional processes that take place in structurally non-homogeneous layers. The development of a method of the theory of functions that satisfy these equations considerably broadens the opportunities for analysing various physical processes.

2. Contrary to the plane-parallel case, we introduce the complex potential for axisymmetric processes in the form

$$W = \varphi + i\psi/y = \varphi - 2\psi/(z - \bar{z})$$

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which in the complex half-plane $z\ (\operatorname{Im} z > 0)$ satisfies the following equation obtained from system (1.2):

$$\frac{\partial W}{\partial \bar z} - \frac{W - \overline{W}}{2(z - \bar z)} = 0 \tag{2.1}$$

Note that the complex conjugate velocity of the process $\bar V = u - iv$ is determined by an equation of the same form (2.1).

The continuous function $W(z)$ of the complex variable z that satisfies Eq.(2.1) in some domain in the half-plane $z\,(\operatorname{Im} z \geqslant 0)$ will be called an axisymmetric generalized analytical function in this domain.

Let us list some fundamental properties of the function $W(z)$ that follow from its definition. By the linearity of Eq.(2.1), its solutions satisfy the superposition principle. A trivial solution of Eq.(2.1) is the generalized constant $A = \alpha - 2\beta/(z - \bar z)$, where α and β are abitrary real constants (this is also the meaning of these symbols in what follows); this solution corresponds, for instance, to a liquid at rest.

An analogoue of Liouville's theorem of the theory of analytical functions for $W(z)$ is stated as follows: if the modulus of $W(z)$ in the entire half-plane $z\ (\operatorname{Im} z \geqslant 0)$, including the point at infinity, is bounded and does not exceed the modulus of a generalized constant $(|W(z)| \leqslant |A|)$, then $W(z)$ is a generalized constant. Hence, it follows that a function $W(z)$ which is not equal to A must have singular points, which are models of physical sources of processes.

Eq.(2.1) is conformally covariant. Among the conformal transformations, it is interesting to consider inversion relative to a sphere of radius a centred at the origin. This transformation is defined as follows: if some function $W_0(z)$ satisfies Eq.(2.1), then

$$W = \frac{a}{2|z|}\int_p^1 \tau^{\sigma-1}\left[2\tau\frac{\partial}{\partial\tau}\overline{W}_0\left(\frac{a^2\tau}{\bar z}\right) + \overline{W}_0\left(\frac{a^2\tau}{\bar z}\right) - W_0\left(\frac{a^2\tau}{\bar z}\right)\right]d\tau \tag{2.2}$$

is also a solution of this equation. The values of p and σ are determined by the form of the function $W_0(z)$: if the singular points of $W_0(z)$ lie outside the sphere and $|W_0(z)| = O(|z|^n)$ as $|z| \to 0$, then $p = 0$, $\sigma > -n$; if there are singular points inside the sphere and $|W_0(z)| = O(|z|^{-n-1})$ as $|z| \to \infty$, then $p = \infty$, $\sigma < n + 1$; here and in what follows, $n = 1, 2, 3, \ldots,$ unless otherwise stated.

3. For the function $W(z)$ which is single-valued and continuous in some simply connected domain D in the half-plane, we introduce the operations of differentiation and integration.

We denote by $d_\Sigma W/dz$ or W_Σ the Σ-derivative of W, which is defined so that everywhere in D it satisfies Eq.(2.1), i.e.,

$$\frac{d_\Sigma W}{dz} \equiv W_\Sigma = \frac{\partial W}{\partial z} + \frac{W - \overline{W}}{2(z - \bar z)} \tag{3.1}$$

Comparing (3.1) with (1.2), we obtain the equality $W_\Sigma = \bar V$, which discloses the physical meaning of W_Σ.

The Σ-integral of W along a piecewise-smooth contour C inside the domain D is defined as

$$\int_C W\, d_\Sigma\zeta = \frac{1}{2}\int_C \left(1 + \frac{\zeta - \bar\zeta}{z - \bar z}\right) W\, d\zeta + \left(1 - \frac{\zeta - \bar\zeta}{z - \bar z}\right)\overline{W}\, d\bar\zeta \tag{3.2}$$

The hydrodynamic interpretation of the Σ-integral follows from the equality for $\bar V$

$$\int_C \bar V\, d_\Sigma\zeta = \Gamma - \Pi/[\pi(z - \bar z)] \tag{3.3}$$

where Γ and Π are the circulation and the velocity flux of the fluid evaluated for the contour C.

Since $W(z)$ satisfies Eq.(2.1), the Σ-integral is independent of the choice of the contour C and is entirely characterized by the position of its initial point z_0 and final point z. Therefore, the Σ-integral viewed as a function of its upper limit z defines the function $W^*(z)$ by the equality

$$\int_{z_0}^{z} W(\zeta)\,d_\Sigma\zeta = W^*(z) - W_0^* \quad (W_0^* = \varphi^*(z_0) - 2\psi(z_0)/(z-\bar z)) \tag{3.4}$$

(W_0^* is the generalized constant), which is an analogue of the Newton-Leibniz formula for analytical functions.

Separating the real and imaginary parts in (3.2) and (3.4), we obtain a standard expression for the integration operator /2, 5/.

For a closed contour C, we obtain from (3.4) the equality

$$\oint_C W(\zeta)\,d_\Sigma\zeta = 0 \tag{3.5}$$

which may be regarded as a generalization of Cauchy's theorem: if $W(z)$ is an axisymmetric generalized analytical function in a simply connected domain D, then the Σ-integral of $W(z)$ over any piecewise-smooth contour C entirely contained in D vanishes.

The converse of the generalized Cauchy theorem may be called Morer's theorem. It provides an alternative definition of the function $W(z)$, which is equivalent to the previous definition based on Eq.(2.1) or to the definition introduced using the condition for the Σ-derivative (3.1) to exist.

Formulas (3.1) and (3.4) are the basis for constructing complex potentials of a process given a known complex potential.

4. Among the solutions of Eq.(2.1) the fundamental solutions that model a ring source (sink) of capacity Q and a ring vortex of total intensity Γ' are of fundamental importance. Their complex potentials, according to /4, 6, 8/, can be represented in the form

$$W = -\frac{Q}{2\pi^2 R_*}\left\{K(k) + i\,\frac{x-x_0}{2y_0}\,[n_1'\Pi(-n_1^2, k) - K(k)]\right\} \tag{4.1}$$
$$W = \frac{\Gamma'}{2\pi^2 R_*}\left\{\frac{x-x_0}{2}\,[n_1'\Pi(-n_1^2, k) - K(k)] - ik^2 C(k)\right\}$$
$$(k = 2\sqrt{yy_0}/R_*,\; n_1^2 = 4yy_0/(y+y_0)^2,\; n_1^2 + n_1'^2 = 1,\; R_* = [(x-x_0)^2 + (y+y_0)^2]^{1/2})$$

where $K(k)$, $C(k)$ and $\Pi(-n_1^2, k)$ are complete elliptic integrals /9/ of modulus k and n_1^2 is a parameter.

Since the complex potential W is defined, apart from an arbitrary generalized constant, this constant may be chosen so that the complex potentials (4.1) are single-valued functions (see /6/). These complex potentials have logarithmic singularities at the point $z_0 = x_0 + iy_0$.

Using the solutions (4.1), we can apply the superposition principle to derive the complex potentials of other axisymmetric processes.

The determination of complex potentials by condensation of singularities may be reduced to Σ-differentiation of the solutions (4.1). We introduce a complex potential of a vortical source with a logarithmic singularity at the point z_0:

$$\ln Z(z, z_0, \alpha, \beta) = \alpha F_1(z, z_0) - i2\beta F_2(z, z_0)/(z_0 - \bar z_0) \tag{4.2}$$

where $F_1(z, z_0)$ and $F_2(z, z_0)$ are normalized complex potentials (4.1) with $Q = 4\pi^2$ and $\Gamma' = 4\pi^2 y_0$.

In what follows, we will use negative formal powers, which may be defined as the n-th order Σ-derivative of (4.2), i.e.,

$$Z^{(-n)}(z, z_0, \alpha, \beta) = \frac{(-1)^n}{(n-1)!}\,\frac{d_\Sigma^{\,n}}{dz^n}\ln Z(z, z_0, \alpha, \beta) \tag{4.3}$$

We then obtain the complex potentials of multipoles with moments M_n:

$$W_n = \frac{M_n}{4\pi^2}(n-1)!\,Z^{(-n)}(z, z_0, \alpha, \beta) \tag{4.4}$$

From (4.1)-(4.4) for $n = 1$, $\alpha = \cos\theta_1$, $\beta = \sin\theta_1$, we obtain the complex potential of a dipole of moment M_1 directed at an angle θ_1 to the x-axis:

$$W_1 = -\frac{M_1 e^{i\theta_1}}{2\pi^2 R^2 R_*}\{[B(k) + k'^2 D(k)e^{-i2\theta_1}][x - x_0 - i(y-y_0)] - i2y_0 k'^2 D(k)e^{-i2\theta_1}\} \tag{4.5}$$
$$(R = [(x-x_0)^2 + (y-y_0)^2]^{1/2},\; k' = \sqrt{1-k^2}).$$

where $B(k)$ and $D(k)$ are complete elliptic integrals of modulus k /9/ and k' is an additional modulus.

For $n = 2, 3, 4, \ldots$, we obtain from (4.1)-(4.4) the corresponding complex potentials of the multipole at the point z_0. If the multipoles are located at the origin $(z_0 = 0)$, then setting $\alpha = 1$, $\beta = 0$, we obtain

$$Z^{(-n)}(z) = r^{-n-1}\left[P_n(\cos\theta) - i\frac{\sin\theta}{n}P_n'(\cos\theta)\right] \tag{4.6}$$

where $P_n(\cos\theta)$ are Legendre polynomials. In this case,

$$W_n = -\frac{M_n}{4\pi}n!\,Z^{(-n)}(z)$$

A positive formal power $Z^{(n)}(z, z_0, \alpha, \beta)$ is defined as the complex potential W_n obtained by n-fold Σ-integration from z_0 to z of the generalized complex constant $\alpha - 2\beta/(z - \bar{z})$ and multiplication by $n!$, i.e.,

$$Z^{(0)}(z, z_0, \alpha, \beta) = \alpha - \frac{2\beta}{(z - \bar{z})}, \quad Z^{(n)}(z, z_0, \alpha, \beta) = n\int_{z_0}^{z} Z_{|}^{(n-1)}(\zeta, z_0, \alpha, \beta)\, d_\Sigma\zeta \tag{4.7}$$

The powers (4.7) define complex potentials of multipoles with n-th order poles at infinity. The form of these powers depends on the choice of α, β, and z_0. In particular, for $\alpha = 1$, $\beta = 0$ and $z_0 = 0$, they take the simplest form

$$Z^{(n)}(z) = r^n\left[P_n(\cos\theta) + i\frac{\sin\theta}{n+1}P_n'(\cos\theta)\right] \quad (n = 0, 1, 2, \ldots) \tag{4.8}$$

5. One of the approaches to the investigation of boundary-value problems relies on a generalization of the Cauchy integral. Using Green's second formula for Eqs.(1.2) and applying the solutions (4.1), we obtain the generalized Cauchy formula

$$\frac{1}{2\pi i}\int_C W(\zeta)\Omega_-(z,\zeta)\,d\zeta - \overline{W}(\zeta)\Omega_+(z,\zeta)\,d\bar{\zeta} = \begin{cases} W(z), & z \in D \\ 0, & z \overline{\in} \bar{D} \end{cases} \tag{5.1}$$

which expresses the axisymmetric generalized analytical function $W(z)$ in terms of its values on the boundary C of the domain D. The integral on the left-hand side is the generalized Cauchy integral, whose kernels

$$\Omega_\pm(z,\zeta) = \frac{\zeta - \bar{\zeta}}{4i}(w_1 \pm w_1')$$

unlike those introduced in /6, 7/, have an obvious physical interpretation: they are expressible in terms of normalized $(M_1 = 4\pi^2)$ complex potentials w_1 and w_1' of the dipoles (4.5) with moments parallel and perpendicular to the x-axis.

Replacing $W(\zeta)$ in (5.1) with a continuous function $f(\zeta)$, we obtain a generalized Cauchy integral, which defines an axisymmetric generalized analytical function $W(z)$. For this function, we can prove an analogue of Sokhotskii's formulas, the analytical continuation theorem, and the reflection principle.

6. A different approach to the solution of boundary-value problems is by expanding the axisymmetric generalized analytical function $W(z)$ in formal power series, which is possible for solving equations of the form (2.1) (see, e.g., /3/).

The function $W(z)$ in a circle of radius R_0 centred at the point z_0 can be represented by the generalized Taylor series in powers (4.7), i.e.,

$$W(z) = \sum_{n=0}^{\infty} Z^{(n)}(z, z_0, \alpha_n, \beta_n) \tag{6.1}$$

where the real coefficients α_n and β_n are determined from the equalities

$$\alpha_0 - \frac{|2\beta_0}{z_0 - \bar{z}_0} = W(z_0), \quad n!\left[\alpha_n - \frac{2\beta_n}{z_0 - \bar{z}_0}\right] = \frac{d_\Sigma{}^n W(z)}{dz^n}\bigg|_{z=z_0}$$

which are obtained by n-fold Σ-differentiation of this series. In particular, if series (6.1) consists of the powers (4.8) and contains either all powers or only odd or even powers, then we obtain functions which are analogues of the corresponding analytical functions: the exponential, the sine, or the cosine functions.

In the ring $(R_1 < |z - z_0| < R_2)$, the function $W(z)$ can be represented as a generalized Laurent series in powers (4.3) and (4.7), i.e.,

$$W(z) = \sum_{n=0}^{\infty} Z^{(n)}(z, z_0, \alpha_n, \beta_n) + \sum_{n=1}^{\infty} Z^{(-n)}(z, z_0, \alpha_{-n}, \beta_{-n}) \tag{6.2}$$

By (3.5) and (6.2), we can assert that $W(z)$ satisfies an equality generalizing the expression of the main theorem on residues of analytical functions, i.e.,

$$\frac{1}{2\pi i}\int_{C_0} W(\zeta)\, d_\Sigma\zeta = \sum_{\nu=1}^{m} \frac{1}{2\pi i}\int_{C_\nu} W(\zeta)\, d_\Sigma\zeta$$

Here C_ν are non-intersecting contours encircling the isolated singularities z_ν of the function $W(z)$ and entirely contained inside the closed contour C_0.

Thus, the foundations of the theory of axisymmetric generalized analytical functions presented above constitute a complete analogue of the basic propositions of the theory of analytical functions and provide a method of analysing dynamic processes.

7. Let us use the method of axisymmetric generalized analytical functions to analyse three-dimensional seepage in a piecewise-homogeneous medium separated by the surface L into zones with permeability coefficients k_1 and k_2, where the flow is described by the complex potentials

$$W_j = k_j\varphi_j - 2\psi_j/(z - \bar{z}) \quad (j = 1, 2)$$

The conditions of continuity for the pressure and fluid flux are satisfied on L. For φ these conditions have the form /1, 10/

$$\varphi_1|_L = \varphi_2|_L, \quad k_1\varphi_{1n}|_L = k_2\varphi_{2n}|_L \tag{7.1}$$

If L is a sphere or a plane, the solution of the problem can be represented in a finite form.

Assume that the flow in the unbounded medium with permeability coefficient k_0 (we take $(k_0 = 1)$ is described by the complex potential $W_0(z) = \varphi_0 - 2\psi_0/(z - \bar{z})$, the singular points of which are located arbitrarily relative to a sphere of radius a centred at the origin. This potential can be represented in the form

$$W_0(z) = W_{01}(z) + W_{02}(z)$$

where the singular points of $W_{01}(z)$ and $W_{02}(z)$ lie inside and outside the sphere, and

$$|W_{01}(z)| = O(|z|^n), \ |z| \to 0; \quad |W_{02}(z)| = O(|z|^{-n-1}), \ |z| \to \infty$$

If the exterior and the interior of the sphere are filled with media with permeability coefficients k_1 and k_2 respectively, then the complex flow potentials inside and outside the sphere are

$$W_1 = W_0(z) + \lambda\left[\frac{a}{2|z|}\int_0^1 \tau^{(1-\lambda)/2} H_1\left(\frac{a^2\tau}{\bar{z}}\right)d\tau + \int_\infty^1 \tau^{-(1+\lambda)/2} G_2(z\tau)\,d\tau\right] \tag{7.2}$$

$$W_2 = W_0(z) - \lambda\left[\frac{a}{2|z|}\int_\infty^1 \tau^{-(1-\lambda)/2} H_2\left(\frac{a^2\tau}{\bar{z}}\right)d\tau + \int_0^1 \tau^{(1-\lambda)/2} G_1(z\tau)\,d\tau\right]$$

$$H_j\left(\frac{a^2\tau}{\bar{z}}\right) = 2\tau\frac{\partial}{\partial\tau}\overline{W}_{0j}\left(\frac{a^2\tau}{\bar{z}}\right) + \overline{W}_{0j}\left(\frac{a^2\tau}{\bar{z}}\right) - W_{0j}\left(\frac{a^2\tau}{\bar{z}}\right)$$

$$G_j(z\tau) = \frac{\partial}{\partial\tau}[\tau W_{0j}(z\tau)] \quad (j = 1, 2), \quad \lambda = \frac{k_1 - k_2}{k_1 + k_2}$$

The solution (7.2) satisfies conditions (7.1). It was obtained by expanding (6.2) in series in powers (4.6) and (4.8) and then generalizing by formula (2.2) to the case of an arbitrary location of the singular points of $W_0(z)$.

In particular, if the medium inside the sphere is impervious $(k_2 = 0$ and therefore $\lambda = 1)$, the solution (7.2) takes the form of Weiss and Butler theorems for a sphere in an axisymmetric ideal fluid flow (see, e.g., /11/). For $k_2 = \infty$ $(\lambda = -1)$, the solution (7.2) describes the flow past a spherical cavern. If the sphere is surrounded by an impervious medium $k_1 = 0$ or by a free fluid $k_1 = \infty$, then (7.2) defines the flow inside the sphere.

Assume that the boundary L is the plane $x = 0$ and the regions $x > 0$ and $x < 0$ are filled by media with permeability coefficients k_1 and k_2. Considering this case as the limit of the previous case, we obtain the solution

$$W_1 = W_0(z) + \lambda[\overline{W_{01}(-\bar{z})} + W_{02}(z)],\ W_2 = W_0(z) - \lambda[W_{01}(z) + \overline{W_{02}(-\bar{z})}] \tag{7.3}$$

Solutions (7.2) and (7.3) are of interest for analysing various dynamic processes described by Eqs.(1.1).

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